Characteristic locus method robustness improvement through optimal static normalizing pre-compensation

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SUMMARY

Controlled systems designed by using the characteristic locus method can be very sensitive to small perturbations in the plant input and output at the frequencies where the plant transfer matrix is far from normal. In order to improve the closed-loop system robustness, previous work proposes the design of a dynamic normalizing pre-compensator followed by the design of a commutative controller for the pre-compensated plant. The restriction on its structure and the need for approximation by a stable rational transfer matrix were among the limitations of the dynamic pre-compensator. This paper shows that it suffices to design a static pre-compensator that makes the pre-compensated plant as closely as possible to a normal matrix in a frequency band containing the crossover frequency. The pre-compensator is found by solving an optimization problem, whose solution is obtained directly by computing either a singular value decomposition of a real matrix or the spectral decomposition of a symmetric matrix, depending on whether normalization is to be achieved in one or more frequencies. Numerical examples illustrate the theoretical results of the paper. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The generalized Nyquist stability criterion [1] is an important tool for stability analysis and design of closed-loop multivariable linear systems since it provides an integrated assessment of closed-loop stability and dynamic performance. The design technique that emerged naturally from the generalized Nyquist stability criterion is the well-known characteristic locus method (CLM) [2–4], and the controllers designed according to the CLM are known as commutative controllers, since they commute with the plant transfer function with respect to multiplication. The commutativity property has the effect of changing a multivariable design in \( m \) single-input–single-output (SISO) designs, where \( m \) is the number of inputs/outputs of the plant to be controlled. This is carried out through the choice of the controller eigenfunctions with the view to modifying the characteristic loci of the plant in the same manner as SISO controllers modify the Nyquist diagram of the plant in scalar systems. In spite of this simplification, the resulting controller is a full multivariable controller, as opposed to those obtained by sequential loop closing or independent design of each loop using

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SISO controllers as is the case of decentralized control systems [5–12].

Despite being acknowledged as an efficient design technique, the CLM was under serious criticism on account of eigenvalue sensitivity [13], i.e. for plants with frequency responses far from normal at a certain frequency band, the characteristic loci can be very sensitive to perturbations in the plant input and output at these frequencies [14, 15]. In order to improve the robustness of closed-loop systems compensated with commutative controllers, a two-stage-design approach has been proposed in [16], as follows: for an \( m \times m \) plant with transfer function \( G(s) \), a dynamic normalizing pre-compensator, i.e. a pre-compensator \( K_p(s) \) that approximately normalizes the pre-compensated plant \( G(jw)K_p(jw) \) at the whole frequency range is obtained initially and, in the sequel, a controller, \( K_c(s) \), that commutes with \( G(s)K_p(s) \) and also internally stabilizes the feedback system [17, 18], is designed. The overall controller transfer function is therefore \( K(s) = K_p(s)K_c(s) \). More recently, with the view to considering the sensitivity of the characteristic loci with respect to perturbations in both the plant input and output, the design of a normalizing pre-compensator for \( G(s) \) has been proposed through the solution of an optimization problem based on a measure of the misalignment between the input and output principal directions of the plant in each frequency [19]. Although this pre-compensation scheme has led to insensitive open-loop characteristic loci, several constraints have been imposed in the pre-compensator structure.

This paper deals with the design of a pre-compensator with the view to normalizing the plant; therefore, improving the robustness of the closed-loop system compensated in accordance with the CLM. It is shown that it suffices to design a static pre-compensator that makes the pre-compensated plant as closely as possible to a normal matrix at a frequency band containing the crossover frequency; therefore, the need for normalization in the whole frequency range, as done in [16,19]. The pre-compensator is found by solving an optimization problem, whose solution is obtained directly by computing either a singular value decomposition of a real matrix or the spectral decomposition of a symmetric matrix, depending on whether normalization is to be achieved in one or more frequencies. In addition, different from previous works, the proposed pre-compensator has no constraints in its structure.

This paper is structured as follows: in Section 2 it is shown that it suffices to use a static pre-compensator that normalizes the plant at the vicinity of the crossover frequency; in Section 3 an optimization problem for the design of a static pre-compensator with the view to normalizing the pre-compensated plant at a desired frequency \( \omega_0 \) is formulated and solved; this optimization problem is also extended to the multi-frequency case in this section; finally, two numerical examples are presented in Section 4 to illustrate the theoretical results presented in the paper.

2. PRELIMINARY RESULTS

Let \( G(s) \) and \( K(s) \) be the \( m \times m \) transfer matrices of the plant and controller, respectively. According to the generalized Nyquist stability criterion [1], the closed-loop system of Figure 1 is stable if and only if the net sum of anticlockwise encirclements of the critical point \(-1+j0\), by the characteristic loci of \( G(s)K(s) \), is equal to the number of unstable poles of \( G(s) \) and \( K(s) \). The use of the generalized Nyquist stability criterion as a design tool is by means of the so-called commutative controllers, that is, a controller \( K(s) \) such that \( G(s)K(s) = K(s)G(s) \). Commutativity is achieved whenever \( G(s) \) and \( K(s) \) share the same eigenvector and dual eigenvector frames, in which case, the eigenvalues of the product \( G(s)K(s) \) are equal to the product of the eigenvalues of \( G(s) \) and \( K(s) \). In this regard, closed-loop stability and performance requirements such as tracking, disturbance rejection and good transient response are achieved through the manipulation of the eigenfunctions of the open-loop transfer matrix, i.e. by choosing, in an adequate

![Figure 1. Closed-loop feedback control system.](image-url)
manner, the controller eigenfunctions. This controller design strategy is known as the CLM. However, as shown in [13], the CLM can only be effective from the robustness point of view if the plant transfer matrix is normal. The definition of normal matrix is as follows.

**Definition 1**
A matrix \( G \in \mathbb{C}^{m \times m} \) is normal if and only if it commutes with its conjugate transpose, \( G^* \), i.e. \( GG^* = G^*G \).

Some measures of approximate normality of a complex matrix are presented in [20]. One of these measures is based on the following theorem.

**Theorem 1**
A matrix \( G \in \mathbb{C}^{m \times m} \) is normal if and only if it has an orthonormal set of \( m \) eigenvectors.

**Proof**
See [21]. \( \square \)

An immediate consequence of Theorem 1 is the definition of the degree of normality of a matrix and of approximately normal matrices. This is done with the help of the following definitions [22].

**Definition 2**
Let \( A \in \mathbb{C}^{m \times m} \). If \( A^{-1} \) exists, then the Euclidean condition number of \( A \), denoted as \( \mathcal{C}(A) \), is given by

\[
\mathcal{C}(A) = \frac{\bar{\sigma}(A)}{\underline{\sigma}(A)}
\]

where \( \bar{\sigma}(\cdot) \) and \( \underline{\sigma}(\cdot) \) denote, respectively, the maximum and minimum singular values of a matrix.

**Definition 3**
Let \( \mathcal{D}^{m \times m} \) be the set of all invertible diagonal matrices in \( \mathbb{C}^{m \times m} \). Under the same assumptions as in Definition 2, then the optimal condition number of a complex matrix \( A \), denoted as \( \mathcal{C}^{opt}(A) \), is defined as:

\[
\mathcal{C}^{opt}(A) = \inf_{D \in \mathcal{D}^{m \times m}} \mathcal{C}(AD)
\]

It is shown in [22] that optimization problem (2) is convex and, therefore, it has only one minimum that can be achieved through convex programming.

Measures of normality and approximately normality can now be introduced, as follows.

\[ v = \mathcal{C}^{opt}(W) - 1 \] (3)

Moreover, when \( v \to 0 \), then the matrix \( G \) is said to be approximately normal.

The optimal condition numbers of the eigenvector matrices of \( G(jw)K(jw) \) and \( K(jw)G(jw) \) can be used to give an upper bound for a measure of robustness of the closed-loop system with respect to perturbations in the plant output and input, respectively. This will be done with the help of the following result.

**Lemma 1 (Small gain theorem)**
Let \( M_G(s) \) be a stable transfer matrix and define

\[
G_p(s) = [I + M_G(s)]G(s)
\]

as the plant transfer matrix with multiplicative perturbation in the plant output, where \( \tilde{\sigma}[M_G(jw)] \leq \mu_G(\omega) \) and \( \mu_G(\omega) \) is a nonnegative real function. Then, \( K(s) \) stabilizes \( G_p(s) \) if and only if \( K(s) \) stabilizes \( G(s) \) and

\[
\tilde{\sigma}[G(jw)K(jw)[I + G(jw)K(jw)]^{-1}] < \frac{1}{\mu_G(\omega)}
\]

\[ \tag{5} \]

**Proof**
See [23]. \( \square \)

A necessary condition for robust stability of the closed-loop system, given in terms of the eigenvalues of the open-loop transfer matrix, can be obtained from Lemma 1 and using the fact that the maximum singular value is an upper bound for the moduli of the eigenvalues of a matrix, as follows:

\[
\max_{i \in \mathcal{I}_m} \left| \frac{\lambda_i[G(jw)K(jw)]}{1 + \lambda_i[G(jw)K(jw)]} \right| < \frac{1}{\mu_G(\omega)}
\]

\[ \tag{6} \]

where \( \mathcal{I}_m = \{1, 2, \ldots, m\} \). Consider now the following result.
Lemma 2
Let \( A \in \mathbb{C}^{m \times m} \) and let \( A = W_A \Lambda_A V_A, \) \( V_A = W_A^{-1} \), be a spectral decomposition of \( A \). Then,
\[
\frac{\tilde{\sigma}[A]}{\epsilon_{\text{op}}(W_A)} \leq \max_{i \in \mathcal{J}_m} |\tilde{\lambda}_i[A]| \tag{7}
\]

Proof
Note that \( A = W_A \Lambda_A V_A = (W_A D) \Lambda_A (D^{-1} V_A) \), for all \( D \in \mathbb{D}^{m \times m} \). Thus
\[
\tilde{\sigma}[A] = \tilde{\sigma}[W_A D \Lambda_A D^{-1} V_A] \leq \frac{\tilde{\sigma}[W_A D]}{\sigma[W_A D]} \tilde{\sigma}[\Lambda_A] = \epsilon(W_A D) \max_{i \in \mathcal{J}_m} |\tilde{\lambda}_i[A]| \tag{8}
\]
Replacing \( D \), in inequality (8), with the optimal solution of problem (2) gives the desired result. \( \square \)

A sufficient condition for robust stability of the closed-loop system, expressed in terms of the eigenvalues of the open-loop transfer matrix, can be obtained with the help of Lemmas 1 and 2, as follows.

Theorem 2
If \( K(s) \) stabilizes \( G(s) \) and for each \( \omega \in \mathbb{R} \)
\[
\max_{i \in \{1,2,\ldots,m\}} \frac{\tilde{\lambda}_i[G(j \omega)K(j \omega)]}{1 + \tilde{\lambda}_i[G(j \omega)K(j \omega)]} \leq \frac{1}{\mu_G(\omega)[1 + v(j \omega)]} \tag{9}
\]
where \( v(j \omega) = \epsilon_{\text{op}}[W(j \omega)] - 1 \) with \( W(j \omega) \) denoting the eigenvector matrix of \( G(j \omega)K(j \omega) \), then \( K(s) \) stabilizes \( G_p(s) \).

Proof
The proof is straightforward and will be omitted. \( \square \)

Remark 1
(i) Similar results to those given in Theorem 2 and inequality (6) can be obtained assuming multiplicative perturbation in the plant input, i.e. \( G_p(s) = G(s) [I + M_G(s)] \). In this case the upper bounds are given in terms of the optimal condition number of the eigenvector matrix of \( K(j \omega)G(j \omega) \).
(ii) When the matrix \( G(s) \) is normal at a certain frequency \( \omega \), then \( v(j \omega) \) becomes zero and thus inequalities (9) and (6) become identical. In this case, inequality (6) provides a necessary and sufficient condition, as claimed in [13].

Based on Lemma 1, Theorem 2 and inequality (6), the following conclusions regarding the normality of the open-loop transfer matrix can be drawn

(1) At high frequencies, since \( G(s) \) is usually strictly proper, \( G(j \omega)K(j \omega) \to 0 \). As a consequence, \( \tilde{\lambda}_i[G(j \omega)K(j \omega)] \to 0 \), \( i = 1, \ldots, m \). Therefore, conditions (6) and (9), become, respectively,
\[
\varepsilon < \frac{1}{\mu_G} \quad \text{and} \quad \varepsilon < \frac{1}{\mu_G(1 + \nu)} \quad \nu \to 0 \tag{10}
\]
which are satisfied for arbitrarily large values of \( \nu \). This implies that there is no need for normalization at high frequencies.

(2) At low frequencies, due to integral action required to track step reference signals, \( G(j \omega)K(j \omega) \) becomes infinite and thus, according to inequality (5), \( \mu_G < 1 \) becomes a necessary and sufficient condition for robust stability of the closed-loop system. Therefore, normalization is not a crucial requirement for low sensitivity of the characteristic loci at low frequencies when integral action is deployed; see also [3].

(3) At the frequencies near the crossover frequency \( \omega_c \), the sufficient condition (9) approaches the necessary condition (6) when \( v \to 0 \), namely that when \( G(j \omega_c)K(j \omega_c) \) becomes approximately normal.

(4) Same conclusions can be drawn for the product \( K(j \omega)G(j \omega) \) concerning perturbations in the plant input. The only difference is that, now, \( \nu \) denotes the degree of normality associated with the optimal condition number of the eigenvector matrix of \( K(j \omega)G(j \omega) \).

Conclusions (1)–(4) ensure that only when \( G(j \omega) \) is not normal at a frequency band containing the crossover frequency \( \omega_c \), a static normalizing precompensator \( K_p \) should be designed in order to make \( G(j \omega)K_p \) approximately normal in this frequency band.
Therefore, instead of designing a dynamic normalizing pre-compensator, as proposed in [16, 19], it suffices to design a static pre-compensator that normalizes the plant at the vicinity of $\omega_c$.

It can be easily proved that although the resulting controller

$$K(s) = K_p K_c(s)$$  \hspace{1cm} (11)

where $K_p$ is the normalizing pre-compensator and $K_c(s)$ is a controller that commutes with $G(s)K_p$ can be designed to improve the closed-loop system robustness with respect to perturbations in the plant output, it may be possible that the closed-loop system robustness with respect to perturbations in plant input be poor. This difficulty can be overcome by introducing another objective on the design of $K_p$, namely that $K_p$ should make both $K_pK_c(jw)G(jw)$ and $G(jw)K_pK_c(jw)$ approximately normal at the frequency range of interest.

3. NORMALIZING PRE-COMPENSATORS

3.1. Problem formulation

With the view to considering the normalization of both $G(s)K(s)$ and $K(s)G(s)$, the so-called reversed-frame-normalizing-controllers has been introduced [20]. It can be seen that both $G(s)K(s)$ and $K(s)G(s)$ are normal if and only if the singular-vector frames of $K(s)$ are those of $G(s)$ taken in reversed order, as stated in the following lemma.

**Lemma 3**
Suppose that $G$, $K \in \mathbb{C}^{m \times m}$ are both of rank $m$ and let

$$G = Y \Sigma U^*$$  \hspace{1cm} (12)

be a singular value decomposition of $G$, where $\Sigma = \text{diag} \{ \sigma_i, i = 1, \ldots, m \}$. Then $GK$ and $KG$ are both normal if and only if

$$K = U \Gamma_K Y^*$$  \hspace{1cm} (13)

for some nonsingular diagonal matrix $\Gamma_K \in \mathbb{C}^{m \times m}$.

**Proof**
See [20].

According to Lemma 3, the characteristic loci of $G(s)K(s)$ are at their least sensitivity to perturbations in the plant input and output if and only if the controller $K(s)$ has the structure given by Equation (13). However, in this paper, the controller is defined by Equation (11), which implies that $K_p$ must have a specific structure so that $G(s)K(s)$ and $K(s)G(s)$ be both normal, as stated below.

**Theorem 3**
Suppose that the product $GK \in \mathbb{C}^{m \times m}$ is of rank $m$ and has all eigenvalues distinct. In addition, assume that all eigenvalues of $GK_p$ are also distinct and let $K = K_pK_c$, where $K_c$ has the same eigenvector matrix as $GK_p$. Then, $GK$ and $KG$ are both normal matrices if and only if

$$K_p = U \Phi Y^*$$  \hspace{1cm} (14)

where $\Phi$ is a nonsingular diagonal matrix.

**Proof**
($\Rightarrow$) If $GK$ and $KG$ are normal matrices, then, according to Lemma 3,

$$K = K_pK_c = U \Gamma_K Y^*$$  \hspace{1cm} (15)

Therefore, from Equations (12) and (13), it can be seen that $GK = Y \Sigma U^* U \Gamma_K Y^* = Y \Sigma \Gamma_K Y^*$, which is a spectral decomposition of $GK$. Since $GK_p$ is assumed to have distinct eigenvalues and $K_c$ and $GK_p$ share the same eigenvector matrices, then $K_c = Y \Lambda_c Y^*$, where $\Lambda_c$ is the eigenvalue matrix of $K_c$, is a spectral decomposition of $K_c$. Using Equation (15), we obtain

$$K_p = U \Gamma_K \Lambda_c^{-1} Y^*$$  \hspace{1cm} (16)

which is equivalent to Equation (14) for $\Phi = \Gamma_K \Lambda_c^{-1}$.

($\Leftarrow$) The proof is straightforward and will be omitted.

Assuming that, for a given frequency $w_0$, $K_c(jw_0)$ is chosen in such a way that all eigenvalues of $G(jw_0)K(jw_0)$ are distinct, then exact normality of $G(jw_0)K(jw_0)$ and $K(jw_0)G(jw_0)$ are achieved through the choice of a nonsingular diagonal matrix $\Phi$ such that $G(jw_0)K_p$ has distinct eigenvalues and by computing $K_p$ in accordance with Equation (14).
However, in this paper, the computation of a normalizing static pre-compensator is considered, which implies that $K_p$ must be a real matrix. Since the matrices $U(j\omega_0)$, $Y(j\omega_0) \in \mathbb{C}^{m \times m}$, then, only for special cases it is possible to obtain $K_p$ real; in [20] the design of static pre-compensators with the structure given by Equation (14) is considered in two cases where it is possible to obtain real pre-compensators, namely that, at d.c. frequency ($\omega_0 = 0$) and at very high frequencies ($\omega_0 \to \infty$). In these cases $U(j\omega_0)$, $Y(j\omega_0) \in \mathbb{R}^{m \times m}$. However, the need for normalization is more critical at frequencies, where the characteristic loci are close to the critical point $-1+j0$, i.e. at intermediate frequencies, as pointed out in Section 2.

Since exact normalities of $G(j\omega)K$ and $KG(j\omega)$ at intermediate frequencies by means of a complex matrix $K = K_pK_c$ ($K_p \in \mathbb{R}^{m \times m}$) cannot, in general, be simultaneously obtained, it is more realistic to seek approximate normality. The following results relates the approximate normality of $G(j\omega)K(j\omega)$ and $K(j\omega)G(j\omega)$ to the approximate normality of $G(j\omega)K_p$ and $K_pG(j\omega)$, respectively.

**Theorem 4**

Let $K(s)$ be given by Equation (11), and assume that $G$ and $K$ denote, respectively, the values of $G(j\omega_0)$ and $K(j\omega_0)$ at a given frequency $\omega_0 \in \mathbb{R}$. In addition, assume that all eigenvalues of $GK_p$ are distinct and $K_c$ has the same eigenvector matrix as $GK_p$. Then (i) $GK$ is approximately normal if $GK_p$ is approximately normal and (ii) assuming that $K_p$ is nonsingular, $KG$ is approximately normal if $K_pG$ is approximately normal.

**Proof**

(i) The proof is straightforward and comes from the fact that since $K_c$ has the same eigenvectors as $GK_p$, then $GK_p$ and $GK$ have the same eigenvector matrices.

(ii) Let a singular value decomposition of $G$ be given by Equation (12) and for a nonsingular matrix $K_p$, define

$$M(j\omega_0) = U(j\omega_0)^*K_pY(j\omega_0)$$

(17)

It follows from Equations (12) and (17) that:

$$GK_p = Y\Sigma U^*UMY^* = Y\Sigma MY^*$$

(18)

Suppose now that $\Sigma = W_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}$, where $\Lambda_{\Sigma M}$ is a diagonal matrix, $W_{\Sigma M}$ is an eigenvector matrix of $\Sigma M$ and $V_{\Sigma M} = W_{\Sigma M}^{-1}$. Then, $GK_p$ can be written as:

$$GK_p = YW_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}Y^*$$

(19)

Since, by assumption, $K_c$ has the same eigenvector matrix as $GK_p$, a spectral decomposition of $K_c$ can always be given as

$$K_c = YW_{\Sigma M}\Lambda_cV_{\Sigma M}Y^*$$

(20)

where $\Lambda_c$ is a diagonal matrix formed with the eigenvalues of $K_c$. Thus, according to Equations (17), (20) and (12), and after simple algebraic manipulations, $KG$ can be written as:

$$KG = UMW_{\Sigma M}\Lambda_c\Lambda_{\Sigma M}V_{\Sigma M}M^{-1}U^*$$

(21)

Therefore, for any commutative controller $K_c$, an eigenvector matrix of $KG$ will be given by $UMW_{\Sigma M}$. In particular, for $K_c = I$, $K_pG$ will have the following spectral decomposition:

$$K_pG = UMW_{\Sigma M}\Lambda_{\Sigma M}V_{\Sigma M}M^{-1}U^*$$

(22)

Since, by assumption, all eigenvalues of $K_pG$ are distinct, then an eigenvector matrix of $K_pG$ is equal to $UMW_{\Sigma M}$ up to a scaling to have the smallest condition number according to Equation (2). Therefore, if the eigenvector matrix of $K_pG$ has optimal condition number approximately equal to one, i.e. if $K_pG$ is approximately normal, then $KG$ is also approximately normal.

Theorems 3 and 4 lead to the formulation of the following optimization problem to find a real pre-compensator $K_p$ that approximately normalizes $G(j\omega)K_p$ and $K_pG(j\omega)$ at $\omega = \omega_0$

$$\min_{(K_p, \Phi)} \|K_p - U\Phi Y^*\|^2_F$$

(23)

where $U = U(j\omega_0)$ and $Y = Y(j\omega_0)$ are obtained according to Equation (12) from a singular value decomposition of $G(j\omega_0)$ and $\|\cdot\|_F$ denotes the Frobenius norm, subject to the following constraints: $K_p \in \mathbb{R}^{m \times m}$ and nonsingular, and $\Phi \in \mathbb{C}^{m \times m}$, diagonal and nonidentically zero.
Optimization problem (23) leads to a pre-compensator that approximately normalizes $G(j \omega)K_p$ and $K_p G(j \omega)$, as shown by the following results.

**Lemma 4 (Bauer and Fike [24])**
Let $A \in \mathbb{C}^{m \times m}$ be a diagonalizable matrix and $A = W \Lambda V$, where $V = W^{-1}$, be a spectral decomposition of $A$ and let $\Delta \in \mathbb{C}^{m \times m}$. In addition, assume that $\hat{\lambda}$ is an eigenvalue of $A + \Delta$. Then there exists an eigenvalue $\lambda_i$ of $A$ such that
\[
|\lambda_i - \hat{\lambda}| \leq \bar{\sigma}(W) \bar{\sigma}(V) \bar{\sigma}(\Delta) = \bar{\epsilon}(W) \bar{\sigma}(\Delta)
\] (24)

**Proof**
See [21]. □

**Lemma 5**
Let $P \in \mathbb{C}^{m \times m}$ be such that $P = e^{i \Psi} + P_0$, where $\Psi$ is a diagonal matrix, $P_0 \in \mathbb{C}^{m \times m}$ and $\bar{\sigma}(P_0) = \epsilon < 1$. Then,
\[
\bar{\epsilon}(P) \leq \frac{1 + \epsilon}{1 - \epsilon}
\] (25)

**Proof**
The proof can be easily obtained using properties of the singular values. □

**Theorem 5**
Let $G = Y \Sigma U^*$ be a singular value decomposition of $G$, $\Phi \in \mathbb{C}^{m \times m}$ be a diagonal matrix and $K_p \in \mathbb{R}^{m \times m}$. Define
\[
E = U^* K_p Y - \Phi
\] (26)
and $Z = \Sigma \Phi = \text{diag}\{z_1, z_2, \ldots, z_m\}$. Then, if $z_i \neq z_\ell$, $i \neq \ell$, $i, \ell \in M$, where $M = \{1, 2, \ldots, m\}$ and
\[
2 \sqrt{m} \|\Sigma\| \|E\| \rightarrow 0
\] (27)
then $v_{G K_p} \rightarrow 0$ and $v_{K_p G} \rightarrow 0$, where $v_{G K_p}$ and $v_{K_p G}$ denote the deviation from normality of $G K_p$ and $K_p G$, respectively, given according to Definition 4.

**Proof**
Let
\[
G_p = G K_p = W_p A_p V_p
\] (28)
be a spectral decomposition of $G K_p$ where $V_p = W_p^{-1}$. Since $Y$ is a unitary matrix, then any eigenvector matrix of $G$ formed with eigenvectors with unity Euclidean norm can be written as
\[
W_p = Y P
\] (29)
where $P \in \mathbb{C}^{m \times m}$ is a nonsingular matrix with all columns with Euclidean norm equal to one. Thus, using Equations (28), (29) and (26), it can be verified that:
\[
\Sigma \Phi + \Sigma E = P \Lambda_p P^{-1}
\] (30)
According to Lemma 4, associated with each diagonal entry of $\Sigma \Phi$, $\sigma_i \phi_i$, there exists an eigenvalue of $\Lambda_p$, $\hat{\lambda}_p$, such that
\[
|\sigma_i \phi_i - \hat{\lambda}_p| \leq \bar{\sigma}(\Sigma E) \leq \|\Sigma\| \|E\| \rightarrow 0
\] (31)
Therefore, defining the complex matrix
\[
\hat{\Lambda} = \Sigma \Phi - \Lambda_p
\] (32)
and using inequality (31) and the fact that $\hat{\Lambda}$ is a diagonal matrix, it follows that:
\[
\bar{\sigma}(\hat{\Lambda}) = \max_i |\sigma_i \phi_i - \hat{\lambda}_p| \leq \|\Sigma\| \|E\| \rightarrow 0
\] (33)
Substituting $\Lambda_p$ given by Equation (32), in Equation (30), and after simple algebraic manipulations, one obtains:
\[
P \Sigma \Phi - \Sigma E P = \Sigma E P + P \hat{\Lambda}
\] (34)
Computing the Frobenius norm of both sides of Equation (34) and using properties of the norm of matrices, it results in:
\[
\|P \Sigma \Phi - \Sigma E P\| \rightarrow 0
\] (35)
Since all columns of $P$ have Euclidean norm equal to one, then $\|P\| \rightarrow \sqrt{m}$. In addition, using Equation (33), one can write
\[
\|P \hat{\Lambda}\| \leq \left( \sum_{i=1}^{m} |\hat{\delta}_i|^2 ||P||^2 \right)^{1/2}
\] (36)
\[
\leq \sqrt{m} \max_i |\hat{\delta}_i| = \sqrt{m} \bar{\sigma}(\hat{\Lambda})
\] (37)
\[
\leq \sqrt{m} \|\Sigma\| \|E\| \rightarrow 0
\] (38)

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where \( \hat{\delta}_i \) denotes the \( i \)th entry of the main diagonal of \( \hat{A} \) and \( p_i \), the \( i \)th column of \( P \). Therefore, inequality (35) can be written as:

\[
\| P\Sigma \Phi - \Sigma \Phi P \|_F \leq 2\sqrt{m} \| \Sigma \|_F \| E \|_F
\]  

(37)

Defining \( Z = \Sigma \Phi \), then, it can be seen that the \((i, \ell)\) entry of \( ZP - ZP \) is given by \((z_i - z_\ell)p_{i\ell} \), where \( p_{i\ell} \) denotes entry \((i, \ell)\) of \( P \). Thus,

\[
\| ZP - ZP \|_F^2 = \sum_{\ell=1}^m \sum_{i=1, i \neq \ell}^m |z_i - z_\ell|^2 |p_{i\ell}|^2
\]  

(38)

Therefore, the following relationship can be established:

\[
\left( \sum_{\ell=1}^m \sum_{i=1, i \neq \ell}^m |p_{i\ell}|^2 \right)^{1/2} \min_{i \neq \ell} |z_i - z_\ell| \leq \| P\Sigma \Phi - \Sigma \Phi P \|_F \leq \sqrt{m} \| E \|_F \| \Sigma \|_F
\]  

(39)

Assuming now that \( E \|_F \) satisfies the condition given by (27), then

\[
\left( \sum_{\ell=1}^m \sum_{i=1, i \neq \ell}^m |p_{i\ell}|^2 \right)^{1/2} \to 0
\]  

(40)

Since all columns of \( P \) have Euclidean norm equal to one, inequality (40) implies that \( P \) is approximately unitary and diagonal. Therefore, according to Lemma 5, \( \| \Phi \| \to 1 \) and since \( \| \Phi(W_p) \| = \| \Phi(Y) \| = \| \Phi(P) \| \geq \| \Phi \|_F \geq \| \Phi \|_{\text{opt}}(P) \), then it is straightforward to see that \( v_{GK_p} \to 0 \).

In order to show that \( v_{K_pG} \to 0 \) when condition (27) is satisfied, it suffices to follow the same steps as the proof above for \( K_pG \); the only difference is that, in this case the eigenvector matrix of \( K_pG \), \( \hat{W}_p \) must satisfy \( \hat{W}_p = UP \), where \( U \) is the input principal direction matrix of a singular value decomposition of \( G \), and \( P \) and \( \hat{W}_p \) have all columns with Euclidean norm equal to one. \( \square \)

Theorem 5 shows that if a pair \((K_p^{\text{opt}}, \Phi^{\text{opt}})\), solution to optimization problem (23), satisfies condition (27) then \( G(j\omega_0)K_p^{\text{opt}} \) and \( K_p^{\text{opt}}G(j\omega_0) \) will both have eigenvector matrices with optimal condition numbers approximately equal to one, being, therefore, approximately normal.

3.2. Solution to the optimization problem

Optimization problem (23) can be written as

\[
\min_{K_p \in \mathbb{R}^{m \times m}, \Phi \text{ nonidentically zero}} J(K_p, \Phi)
\]  

(41)

where

\[
J(K_p, \Phi) = \| K_p - U\Phi \Sigma^* \|_F^2
\]  

(42)

The solution to this problem is given by the following theorem.

Theorem 6

Let \( U_R \) \((U_I)\) and \( Y_R \) \((Y_I)\) denote the real (imaginary) parts of \( U \) and \( Y \), respectively, for a given frequency \( \omega_0 \) and \( u_{R_i} \( (u_{I_i}) \), \( i = 1, 2, \ldots, m \) denote the \( i \)th column of \( U_R \) \((U_I)\) and \( y_{R_i} \( (y_{I_i}) \) denote the \((i, j)\) entry of \( Y_R \) \((Y_I)\). Define the matrix

\[
A = [A_{RR} + A_{II} \ A_{RI} - A_{IR}]
\]  

(43)

where \( A_{RR}, A_{II}, A_{RI} \) and \( A_{IR} \) are defined as

\[
A_{z\beta} = \begin{bmatrix}
u_{x_1}y_{\beta_1} & \nu_{x_2}y_{\beta_1} & \cdots & \nu_{x_m}y_{\beta_m} \\
u_{x_1}y_{\beta_2} & \nu_{x_2}y_{\beta_2} & \cdots & \nu_{x_m}y_{\beta_2} \\
\vdots & \vdots & \ddots & \vdots \\
u_{x_1}y_{\beta_m} & \nu_{x_2}y_{\beta_m} & \cdots & \nu_{x_m}y_{\beta_m}
\end{bmatrix}
\]  

(44)

with \( z \) and \( \beta \) being replaced with \( R \) or \( I \) where appropriate. In addition, write

\[
K_p = [k_1 \ k_2 \ \ldots \ k_m]
\]  

(45)

where \( k_i \) denotes the \( i \)th column of \( K_p \) and \( \Phi = \Phi_R + j\Phi_I \), where \( \Phi_R \) and \( \Phi_I \) are diagonal matrices with real entries. Form the vectors

\[
\phi = [\phi_R^T \ \phi_I^T]^T
\]  

(46)

\[
k = [k_1^T \ k_2^T \ \ldots \ k_m^T]^T
\]  

(47)

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where $\phi_R$ and $\phi_I$ are column vectors formed with the diagonal entries of $\Phi_R$ and $\Phi_I$. Let $A = Y_A \Sigma_A U_A^*$ be a singular value decomposition of $A$, $\sigma_A$ be the largest singular value of $A$ and $u_{A1} = (Y_A)$ the first column of $U_A$ ($Y_A$). Assuming that $\phi$ is normalized to have unit Euclidean norm, then $\min \|K_p - U \Phi Y^*\|_F^2 = 1 - \sigma_A^2$, and is achieved by the pair $(K_p^{\text{opt}}, \Phi^{\text{opt}})$ obtained from the entries of $\phi^{\text{opt}} = u_{A1}$ and

$$k^{\text{opt}} = \sigma_A y_{A1}$$

(48)

Proof

After some simple algebraic manipulations, $J(K_p, \Phi)$ can be written as

$$J(K_p, \Phi) = \text{tr}(K_p K_p^T) - 2 \Re[\text{tr}(U^* K_p Y \Phi^*)] + \text{tr}(\Phi^* \Phi)$$

(49)

where $\Re\{\cdot\}$ and $\text{tr}(\cdot)$ denote, respectively, the real part and the trace of a complex matrix. Since, $U = U_R + j U_I$, $Y = Y_R + j Y_I$ and $\Phi = \Phi_R + j \Phi_I$, then $J(K_p, \Phi)$ can also be written as

$$J(K_p, \Phi) = \text{tr}(K_p K_p^T) - 2 H + \text{tr}(\Phi^* \Phi)$$

(50)

where

$$H = \text{tr}(U_R^T K_p Y_R \Phi_R) + \text{tr}(U_I^T K_p Y_I \Phi_I) - \text{tr}(U_I^T K_p Y_R \Phi_I) + \text{tr}(U_I^T K_p Y_I \Phi_R)$$

(51)

Notice that each term on the right-hand side of Equation (51) has the same structure, i.e. $\text{tr}(U_1 Y_1 \Phi_1)$, where $\alpha$, $\beta$ and $\gamma$ are replaced with $R$ or $I$, when appropriate. Defining the vectors $\phi$ and $k$, according to Equations (46) and (47), then it is straightforward to check that

$$H = k^T A \phi$$

(52)

where $A$ is defined according to Equation (43). Therefore, the problem of minimizing $J(K_p, \Phi)$ given by Equation (50) is equivalent to the problem of minimizing

$$J(k, \phi) = k^T k - 2 k^T A \phi + \phi^T \phi$$

(53)

Note that if one fixes the values of the elements of vector $\phi$, then the cost function $J(k, \phi)$ describes a paraboloid whose minimum value is given by

$$J_{\min}(\phi) = \phi^T \phi - k^T k$$

(54)

where $k = A \phi$. Therefore, the problem of minimizing $J(k, \phi)$, given by Equation (53), is equivalent to the following optimization problem:

$$\min_{\phi} \phi^T (I - A^T A) \phi$$

(55)

for all $\phi$ such that $\Phi$ is nonidentically zero. Note that since the cost function of problem (55) is equivalent to the norm defined in Equation (42), then the matrix $(I - A^T A)$ is either positive definite or positive semidefinite. Furthermore, since the Euclidean norm of vector $\phi$ is assumed to be equal to one, it can be seen that problem (55) is equivalent to:

$$\max_{\phi} \phi^T (A^T A) \phi$$

(56)

Let $A = Y_A \Sigma_A U_A^*$ be a singular value decomposition of $A$. Then, the solution to optimization problem (56) is achieved for $\phi^{\text{opt}} = u_{A1}$, the first column of $U_A$, and the maximum value of cost function (56) is given by $\sigma_A^2$ (the square of the largest singular value of $A$). Furthermore, it is not hard to check that $k^{\text{opt}} = A \phi^{\text{opt}} = \sigma_A y_{A1}$, where $y_{A1}$ denotes the first column of $Y_A$. □

Remark 2

In general, choosing $k = \sigma_A y_{A1}$ leads to a nonsingular $K_p$; this has actually happened in all examples considered so far [25]. However, if it happens that the minimum is achieved with a singular $K_p$, then a nonsingular $K_p$ must be obtained among the suboptimal solutions of problem (56). In order to do so in a systematic manner, note that any vector $\phi$ can always be written as a linear combination of the columns of the matrix $U_A$, as follows:

$$\phi = x_1 u_{A1} + x_2 u_{A2} + \cdots + x_{2m} u_{A2m}$$

(57)

where it is straightforward to see that $x_1 = 1, x_2 = x_3 = \cdots = x_{2m} = 0$ for $\phi^{\text{opt}}$. Therefore, the cost of problem (56) can be written in terms of $x_i, i = 1, 2, \ldots, 2m$, as

$$\phi^T (A^T A) \phi = x_1^2 \sigma_{A1}^2 + x_2^2 \sigma_{A2}^2 + \cdots + x_{2m}^2 \sigma_{A2m}^2$$

(58)

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with the constraint
\[ x_1^2 + x_2^2 + \cdots + x_{2m}^2 = 1 \] (59)
which is implied by \( \| \phi \|_2^2 = 1 \). Moreover, since \( k = A\phi \), then using Equation (57), we may write
\[ k = x_1 \sigma_{A_1} y_{A_1} + x_2 \sigma_{A_2} y_{A_2} + \cdots + x_p \sigma_{A_p} y_{A_p} \] (60)
where \( y_{A_i} \), \( i = 1, \ldots, p \), are the first \( p \) columns of the output principal direction matrix \( Y_A \) associated with nonzero singular values. Using Equation (60), it is possible to express \( K_p \) as
\[ K_p = z_1 \sigma_{A_1} Y_{A_1} + z_2 \sigma_{A_2} Y_{A_2} + \cdots + z_p \sigma_{A_p} Y_{A_p} \] (61)
where \( Y_{A_i} \) is a \( m \times m \) matrix formed from \( y_{A_i} \), \( i = 1, 2, \ldots, p \). Let \( \sigma^2 \) denote the value of cost (58). Therefore, it can be easily checked that
\[ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_p^2}{a_p^2} = 1 \] (62)
where \( a_i = \sigma / \sigma_{A_i} \), \( i = 1, 2, \ldots, p \). Note that Equations (59) and (62) define, respectively, a hypersphere of radius 1 and a hyper-ellipsoid of semi-axes \( a_i \), \( i = 1, 2, \ldots, p \). Furthermore, the following can be stated about their interception points (it is assumed here, for simplicity, that \( A \) has distinct singular values): (i) when \( \sigma = \sigma_{A_i} \), then \( a_1 = 1 \) and \( a_i > 1 \), \( i = 2, 3, \ldots, p \), the hypersphere is inside the hyper-ellipsoid and is tangent to it at point \((1,0,\ldots,0)\); (ii) as \( \sigma \) decreases, \( a_1 \) becomes smaller than 1, which creates several interception points; (iii) when \( \sigma = \sigma_p \), the hyper-ellipsoid becomes internal and tangent to the hypersphere and the unique interception point is \((0,0,\ldots,1)\). Therefore, a nonsingular \( K_p \) that leads to a suboptimal solution to optimization problem (56) can be found using a search algorithm.

It may happen that the solution to optimization problem (55) for just one frequency leads to a pre-compensated plant that is approximately normal only at a small vicinity of the frequency \( \omega_0 \) considered in the optimization problem. If a larger band is required, then it is necessary to consider more than one frequency in the computation of the pre-compensator \( K_p \). This problem is addressed in the next section.

### 3.3. Extension to multi-frequencies

Assume that \( l \) distinct frequencies are chosen in the intermediate frequency range. Then, the cost function (42) can be modified in order to compute a static pre-compensator, as follows:
\[ J(K_p, \Phi_n) = \sum_{n=1}^{l} \bar{w}_n \| K_p - U_n \Phi_n Y_n^* \|_F^2 \] (63)
where \( \{\bar{w}_n \in \mathbb{R}^+, n = 1, \ldots, l\} \) is a set of weights, \( K_p \in \mathbb{R}^{m \times m} \), \( \Phi_n \in \mathbb{C}^{m \times m} \) is a diagonal matrix, and \( U_n, Y_n \) are, respectively, the input and output principal direction matrices of a singular value decomposition of \( G(j\omega_n) \) for \( n = 1, \ldots, l \). It is not hard to check that if \( K_p^{\text{opt}} \) and \( \Phi_n^{\text{opt}} \), \( n = 1, 2, \ldots, l \), satisfy condition (27) then \( G(j\omega_n)K_p^{\text{opt}} \) and \( K_p^{\text{opt}}G(j\omega_n) \) will be approximately normal at the selected frequencies \( \omega_n \).

The problem of minimizing \( J(K_p, \Phi_n) \) can be converted to an eigenvalue problem, whose solution \( (K_p^{\text{opt}}, \Phi_n^{\text{opt}}, \Phi_n^{\text{opt}}, \ldots, \Phi_n^{\text{opt}}) \) is obtained according to the following Theorem.

**Theorem 7**

Let \( U_{nR} \) (\( U_{nI} \)) and \( Y_{nR} \) (\( Y_{nI} \)) denote the real (imaginary) parts of \( U_n \) and \( Y_n \), respectively, for a given frequency \( \omega_n \) and \( u_{nRi}, u_{nRj}, (u_{nRi}) \), \( i = 1, 2, \ldots, m \), denote the \( i \)th column of \( U_{nR} \) (\( U_{nI} \)) and \( y_{nRij}, (y_{nRij}) \) denote the \( (i, j) \) element of \( Y_{nR} \) (\( Y_{nI} \)). Define the matrix
\[ A_n = [A_{nR} + A_{nI} A_{nR} - A_{nI}] \] (64)
where \( A_{nR}, A_{nI}, A_{nR}, \text{ and } A_{nI} \) are defined as
\[ A_{nR} = \begin{bmatrix} u_{nR1} & u_{nR2} & \cdots & u_{nRm} & y_{nR1} & y_{nR2} & \cdots & y_{nRm} \\ u_{nR2} & u_{nR1} & \cdots & u_{nRm} & y_{nR2} & y_{nR1} & \cdots & y_{nRm} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{nRm} & u_{nRm} & \cdots & u_{nR1} & y_{nRm} & y_{nRm} & \cdots & y_{nRm} \end{bmatrix} \] (65)
with \( z \) and \( \beta \) being replaced with \( R \) or \( I \), where appropriate, and form the following matrix:
\[ \tilde{A} = [A_1 \ A_2 \ \ldots \ A_l] \] (66)
In addition, form the vector
\[ \tilde{\Phi} = [\tilde{w}_1 \Phi_n^T \tilde{w}_2 \Phi_n^T \ldots \tilde{w}_l \Phi_n^T]^T \] (67)
where
\[ \Phi_n = [\Phi_n^R \Phi_n^I]^T, \quad n = 1, 2, \ldots, l \] (68)
with \( \Phi_n^R \) and \( \Phi_n^I \) being column vectors formed with the diagonal entries of \( \Phi_n^R \) and \( \Phi_n^I \) (\( \Phi_n^R \) and \( \Phi_n^I \) are diagonal matrices corresponding to the real and imaginary parts of \( \Phi_n \)). Write
\[ K_p = [k_1 \quad k_2 \quad \ldots \quad k_m] \] (69)
and stack its columns to form the following vector:
\[ \tilde{k} = [k_1^T \quad k_2^T \quad \ldots \quad k_m^T]^T \] (70)
Finally, let \( \lambda_{p_1} \) and \( v_{p_1} \) denote the smallest eigenvalue and the corresponding eigenvector of the matrix
\[ P = \text{diag}\left\{ \frac{1}{w_1} I_{2m}, \frac{1}{w_2} I_{2m}, \ldots, \frac{1}{w_l} I_{2m} \right\} - \frac{1}{\sum_{n=1}^l \tilde{w}_n} \tilde{A}^T \tilde{A} \] (71)
where \( I_{2m} \) denote the identity matrix of order \( 2m \).
Assuming that \( \hat{\phi} \) is normalized to have unit Euclidean norm, then \( \min \sum_{n=1}^l \tilde{w}_n \| K_p - U_n \Phi_n Y_n^* \|_{\hat{\phi}}^2 = \lambda_{p_1} \) and is achieved by the \((l+1)\)-tuple \((K_p^{\text{opt}}, \Phi_1^{\text{opt}}, \Phi_2^{\text{opt}}, \ldots, \Phi_{l}^{\text{opt}})\) obtained from the entries of \( \hat{\phi}^{\text{opt}} = v_{p_1} \) and
\[ k_{\text{opt}} = \frac{1}{\sum_{n=1}^l \tilde{w}_n} \tilde{A} v_{p_1} \] (72)
\[ \lambda_{p_1} = \min_{\hat{\phi}} \frac{\hat{\phi}^T P \hat{\phi}}{\hat{\phi}^T \hat{\phi}} \] (76)
where \( P \) is defined in accordance with Equation (71).
Note that, since minimization problems (76) and (63) are equivalent, then the matrix \( P \) is either a positive definite or a semi-definite matrix. Thus, assuming that \( \hat{\phi} \) has Euclidean-norm equal to one, then the minimum of Equation (76) is achieved when \( \hat{\phi} = v_{p_1} \), where \( v_{p_1} \) denotes the eigenvector associated with the smallest eigenvalue, \( \lambda_{p_1} \), of \( P \). Moreover, the minimum is given by \( \lambda_{p_1} \). \( \square \)

4. NUMERICAL EXAMPLES

In this section, two examples will be used to illustrate the pre-compensator design proposed in this paper: the first example illustrates the improvement that can be
achieved by considering more than one frequency in the optimization problem; the second example addresses the design of both the normalizing pre-compensator and the commutative controller.

4.1. Example 1
Consider the transfer matrix of a linearized model of the vertical plane dynamics of an aircraft [26] given by

\[ G(s) = \frac{1}{d(s)} N(s) \]

where \( d(s) = s^5 + 1.5953s^4 + 1.7572s^3 + 0.1112s^2 + 0.0561s \), \( N(s) = [n_{ij}(s)] \), and

\[
\begin{align*}
n_{11}(s) &= -1.5705s^3 - 1.190s^2 \\
&\quad + 1.5409s - 0.0816 \\
n_{12}(s) &= 0.2909s^2 + 0.2527s + 0.3712 \\
n_{13}(s) &= 0.0732s^3 - 0.0646s^2 \\
&\quad - 1.2125s - 0.0204 \\
n_{21}(s) &= -0.12s^4 - 0.0739s^3 \\
&\quad - 0.5319s^2 - 0.2458s \\
n_{22}(s) &= s^4 + 1.5415s^3 + 1.6537s^2 \\
n_{23}(s) &= -0.0052s^3 + 0.1570s^2 + 0.1828s \\
n_{31}(s) &= 4.419s^3 + 1.6674s^2 + 0.1339s \\
n_{32}(s) &= 0.0485s^2 + 0.3279s \\
n_{33}(s) &= -1.6650s^3 - 1.1574s^2 - 0.0918s
\end{align*}
\]

Let \( \omega_c = 10 \text{ rad/s} \) [26] be the desired crossover frequency. As can be seen from Figure 2(a) (solid line), the condition number of the eigenvector matrix of \( G(j\omega) \) is large for almost all frequencies and approximately equal to seven for \( G(j10) \), which shows that \( G(s) \) is not normal at the vicinity of \( \omega_c \). Therefore, it is necessary to compute a normalizing pre-compensator in order to normalise both \( GK_p \) and \( K_p G \) at the vicinity of \( \omega_c \). This can be done as follows: (i) form matrix \( A \), according to Equation (43), and compute its singular value decomposition; (ii) compute the largest singular value of \( A \) and its corresponding output principal direction, \( y_{A1} \); (iii) obtain \( k_{opt} \) in accordance with Equation (48) and form \( K_p \) using Equations (45) and (47). Proceeding in this way, we obtain the following normalizing pre-compensator:

\[
K_{p1} = \begin{bmatrix}
0.3171 & 0.0030 & 0.0731 \\
0.0385 & -0.0138 & 0.0080 \\
0.8953 & 0.0027 & 0.3010
\end{bmatrix}
\]

for which, the minimum value of the cost function (55) is equal to 0.000142463442298. From Figure 2(a) (dashed line), it can be seen that \( G(j\omega)K_{p1} \) is closer to normal than \( G(j\omega) \) at almost all frequencies. Thus, according to Theorem 2, the characteristic loci of the open-loop system can be used to give a reliable measure of robust stability as far as uncertainty in the plant output is concerned. However, as shown in Figure 2(a) (dash-dotted line), the optimal condition number of \( K_{p2}G(j\omega) \) at \( \omega = 1 \text{ rad/s} \) is approximately 6.58 and thus the characteristic loci can be sensitive to model uncertainties in the plant input around this frequency. This shows the need for considering the multi-frequency approach of Section 3.3, which is carried out as follows: (i) choose two frequencies \( \omega_1 = 1 \text{ rad/s} \) and \( \omega_2 = 10 \text{ rad/s} \), and, as weights, \( \tilde{w}_1 = \tilde{w}_2 = 1 \); (ii) form, according to Equation (71), matrix \( P \) (a 12 \times 12 dimensional matrix); (iii) compute the smallest eigenvalue of \( P (\lambda_{P1}) \), and the corresponding eigenvector \( y_{P1} \); (iv) obtain \( k_{opt} \) according to Equation (72), and \( K_p \) according to Equations (70) and (69). Proceeding this way, the following pre-compensator is obtained:

\[
K_{p2} = \begin{bmatrix}
0.1851 & -0.0817 & -0.9711 \\
-0.0683 & 0.5303 & -0.1237 \\
-0.7982 & -0.1022 & 0.0072
\end{bmatrix}
\]

for which the value of the cost function (63) is 0.0335. The optimal condition numbers of the eigenvector matrices of \( G(j\omega)K_{p2} \) and \( K_{p2}G(j\omega) \) are shown in Figure 2(b) (dashed and dash-dotted lines, respectively). It can be seen that, now, both matrices are approximately normal at the frequency range of interest (the optimal condition numbers of the eigenvector
Figure 2. (a) Optimal condition numbers of the eigenvector matrices of $G(j\omega)$ (solid line), $G(j\omega)K_p$ (dashed line) and $K_pG(j\omega)$ (dash-dotted line); and (b) optimal condition numbers of the eigenvector matrices of $G(j\omega)$ (solid line), $G(j\omega)K_p$ (dashed line) and $K_pG(j\omega)$ (dash-dotted line).

matrices are smaller than 1.5 for all frequencies greater than 0.8 rad/s. Therefore, the CLM can now be effectively applied to the pre-compensated plant assuming uncertainties at both the plant input and output.

4.2. Example 2

The plant to be considered here is that proposed in [13], for which the CLM is known to be very sensitive to uncertainties:

\[
G(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} -47s + 2 & 56s \\ -42s & 50s + 2 \end{bmatrix}
\]  \hfill (79)

It is suggested in [13] that, since the eigenfunctions of $G(s)$ are equal to $\hat{\lambda}_{g1}(s) = 1/(s+1)$ and $\hat{\lambda}_{g2}(s) = 2/(s+2)$, then appropriate gain and phase margins can be achieved with the commutative controller $K(s) = I$. However, for this controller, the maximum singular value of the closed-loop transfer matrix $T_c(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$ at $\omega \approx 3$ rad/s is very high (approximately 16.2), showing that the closed-loop system is very sensitive to plant multiplicative uncertainties at the vicinity of this frequency, i.e. even for small variations on $G(s)$, the characteristic loci of the perturbed system differ a great deal from those of the nominal system (the perturbed system may actually become unstable). The low tolerance to plant multiplicative uncertainty can be explained by the lack of normality of $G(s)$ at almost all frequencies: the optimal condition number of the eigenvector matrix of $G(j\omega)$ is equal to 196 at all frequencies, except at the dc frequency, when $G(s)$ becomes equal to the identity matrix. It is therefore necessary to design a normalizing pre-compensator for $G(s)$.

Suppose that the desired bandwidth frequency is $\omega_b = 1$ rad/s, and thus, the characteristic loci should be insensitive to perturbations in the plant input and output in the vicinity of $\omega_b$. Following the same steps as in the first part of Example 1, the following optimal static pre-compensator is obtained:

\[
K_p = \begin{bmatrix} 0.0216 & -0.7068 \\ 0.7068 & 0.0216 \end{bmatrix}
\]  \hfill (80)

It can be easily verified that the optimal condition numbers of the eigenvector matrices of $G(j\omega)K_p$ and $K_pG(j\omega)$, are both equal to one at all frequencies, showing that $G(j\omega)K_p$ and $K_pG(j\omega)$ are both approximately normal matrices.
Consider now the design of a commutative controller for $G_p(s) = G(s)K_p$ and assume the following design specifications: (S1) rise time smaller than 5 s for each loop; (S2) low interactions between outputs; (S3) good damping of step responses with peak overshoot no greater than about 10%; (S4) zero steady-state error for step reference input and disturbance; (S5) tolerance to multiplicative uncertainty of size up to $\frac{1}{1.2}$.

Deploying the degrees of freedom available in the parametrization of all rational commutative controllers [18], a controller $K_c(s)$ that commutes with $G_p(s) = G(s)K_p$ can be obtained. The details involved in the calculation of $K_c(s)$ are not presented here since the design of commutative controllers is not the paper main focus; the interested reader is referred to [18]. One of such controllers, has the following transfer function:

$$K_c(s) = \frac{1}{dK_c(s)} \begin{bmatrix} 138.4s^4 + 1790s^3 + 4210.1s^2 + 2036.4s + 2 \\ -154.9s^4 - 2009.7s^3 - 4792.4s^2 - 2467.6s - 93.4 \\ -154.9s^4 - 1996.8s^3 - 4627.4s^2 - 2080.6s + 93.4 \\ 173.5s^4 + 2243.8s^3 + 5277.1s^2 + 2551.5s + 2 \end{bmatrix}$$
Figure 4. Moduli of eigenfunctions (dashed and solid lines) and maximum singular value (circles) of the closed-loop transfer matrix.

where \( d_K(s) = s^5 + 22.3s^4 + 143.3s^3 + 210.4s^2 + 82.6s \), and has the following properties:

1. It commutes with \( G_p(s) \) at all frequencies, as it can be verified by the high degree of commutativity between \( G_p(j\omega) \) and \( K_c(j\omega) \), according to the commutativity indicator

\[
e(\omega) = \frac{|\lambda Q_i(j\omega) - \lambda G_{p_i}(j\omega)\lambda K_{c_i}(j\omega)|}{|\lambda Q_i(j\omega)|} \times 100% \quad i = 1, 2
\]

which is smaller than \( 2.5 \times 10^{-3} \% \) for \( i = 1, 2 \), where \( Q(s) = G_p(s)K_c(s) \).

2. It satisfies design specifications S1–S4, as one can see from Figure 3.

3. It also satisfies design specification S5, as one can conclude from Figure 4. In this figure, it is shown the moduli of the eigenfunctions (dashed and solid lines) and the maximum singular value (circles) of the closed-loop transfer matrices \( T_c(s) = G(s)K(s)[I + G(s)K(s)]^{-1} \); the moduli of the eigenfunctions and the maximum singular value of \( T'_c(s) = [I + K(s)G(s)]^{-1}K(s)G(s) \) for the frequency range of interest have not been shown, but it can be easily checked that they coincide with those of the closed-loop transfer function. In addition, note that the largest singular value and the largest modulus of the eigenfunctions are approximately equal for the whole frequency range showing that the pre-compensation scheme proposed here has actually succeeded in providing the basis for the design of a reliable commutative controller, as far as sensitivity to uncertainty in both the plant input and output is concerned.

5. CONCLUSIONS

In this paper, it is shown that, to improve the robustness of a multivariable system compensated with a controller designed in accordance with the CLM, the first step is to pre-compensate the plant transfer matrix with a static pre-compensator that makes the pre-compensated system approximately normal in a frequency band containing the crossover frequency.
In order to achieve this goal, a systematic methodology for the design of an optimal static normalizing pre-compensator is presented in the paper. The effectiveness of the proposed pre-compensation scheme is illustrated by means of two numerical examples taken from the open literature.

REFERENCES


